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Abstract

In section 1 we present a general principle for associating nonlinear equations of evolutions with linear operators so that the eigenvalues of the linear operator are integrals of the nonlinear equation. A striking instance of such a procedure is the discovery by Gardner, Miura and Kruskal that the eigenvalues of the Schrödinger operator are integrals of the Korteweg-de Vries equation.

In section 2 we prove the simplest case of a conjecture of Kruskal and Zabusky concerning the existence of double wave solutions of the Korteweg-de Vries equation, i.e. of solutions which for $|t|$ large behave as the superposition of two solitary waves travelling at different speeds. The main tool used is the first of a remarkable series of integrals discovered by Kruskal and Zabusky.

INTEGRALS OF NONLINEAR EQUATIONS OF EVOLUTION AND SOLITARY WAVES

In this paper we study the equation

$$u_t + uu_x + u_{xxx} = 0 \tag{1.1}$$

introduced by Korteweg and de Vries in their approximate theory of water waves, [3]; we shall refer to it as the KdV equation. Subsequently the KdV equation was found to be relevant for the description of hydromagnetic waves [2], and in the description of acoustic waves in an anharmonic crystal, [8]. (1.1) is a special instance of a <u>nonlinear evolution equation</u> of form

$$u_t = K(u). \tag{1.2}$$

We shall study C^{∞} solutions of (1.1) defined for all $x \to \pm\infty$, together with all their x derivatives. It is easy to show that such solutions are <u>uniquely determined by their initial values</u>; for let v be another solution of (1.1):

$$v_t + vv_x + v_{xxx} = 0 \tag*{$(1.1)_v$}$$

Subtracting this from (1.1) and denoting $u - v$ by w we obtain the linear equation

$$w_t + uw_x + wv_x + w_{xxx} = 0$$

for w. Multiplying by w and integrating with respect to x

over $(-\infty,\infty)$ we obtain, after integration by parts, and using the fact that w and w_{xx} tend to zero as $x \to \pm\infty$ the relation

$$\frac{d}{dt}\frac{1}{2}\int w^2 dx + \int (v_x - \tfrac{1}{2}u_x) w^2 dx = 0 . \tag{1.3}$$

Denoting $1/2 \int w^2 dx$ by $E(t)$ and $\operatorname{Max}|2v_x - u_x|$ by m we obtain from (1.3) the inequality

$$\frac{d}{dt} E(t) \leq mE(t).$$

This differential inequality implies that

$$E(t) \leq E(0)e^{mt} \tag{1.4}$$

which implies in particular that if $E(0)$ is zero then so is $E(t)$ - and thereby w - for all t. Furthermore, Sjoberg has shown, [6], that (1.1) has a solution with arbitrarily prescribed initial value $f(x)$ provided that f is smooth enough and tends to zero with sufficient rapidity as $|x|$ tends to infinity.

Equation (1.1) has travelling wave solutions, i.e. of the form $u(x,t) = s(x-ct)$, c being the speed of the wave. To see this substitute into (1.1):

$$- cs_x + ss_x + s_{xxx} = 0 . \tag{1.5}$$

Integrating with respect to x and imposing the boundary condition that s and its derivatives vanish at $x = \pm\infty$

we get

$$(1.6) \qquad -cs + \frac{s^2}{2} + s_{xx} = 0.$$

Multiplication by $2s_x$ and one more integration gives

$$(1.7) \qquad -cs^2 + \frac{s^3}{3} + s_x^2 = 0.$$

From this relation s can be determined explicitly:

$$(1.8) \qquad s(x) = 3c\ \mathrm{sech}^2\left(\frac{x\sqrt{c}}{2}\right)$$

Thus we see that (1.5) <u>has a solution vanishing at</u> $x = \pm\infty$ <u>for every positive speed</u> c, <u>uniquely determined except for a shift which can be so chosen that the maximum of</u> s <u>occurs at</u> $x = 0$. <u>We denote this normalized</u> s, <u>explicitly described by</u> (1.8), <u>as</u> $s(x;c)$; $s(x,c)$ <u>is symmetric in</u> x, <u>decays exponentially as</u> $|x| \to \infty$, <u>and</u> $s(0,c) = 3c$.

On account of its shape s is called a <u>solitary wave</u>.

For linear equations it often happens that all solutions can be obtained as superpositions of a family of special solutions; e.g. all solutions of linear equations with constant coefficients are superpositions of exponential solutions. For nonlinear equations one cannot in general form new solutions out of old and so special families of solutions are not expected to play any special role in the description of all solutions. It was therefore very surprising when Kruskal and Zabusky observed, by analyzing numerically

computed solutions, that all solutions of the KdV equation have hidden in them solitary waves. A precise formulation of this observation is as follows:

Let u be any solution of (1.1) which is defined for all x and t and which vanishes at $x = \pm\infty$. Then there is a discrete set of positive numbers $c_1, \ldots, C_N$ - called the eigenspeeds of u - and sets of phase shifts $\theta_j^{\pm}$ such that

$$\lim_{t\to\pm\infty} u(x+ct,t) = \begin{cases} s(x-\theta_j^{\pm}, c_j) & \text{if } c = c_j \\ 0 & \text{if } c \neq c_j \end{cases} \tag{1.9}$$

Note: Consider any equation of evolution (1.2) which does not involve x and t explicitly, i.e. whose set of solutions is invariant under translation with respect to x and t. Suppose $u(x,t)$ is a solution of such an equation, and that for a certain value of c

$$\lim_{t\to+\infty} u(x+ct,t)$$

exists uniformly on compact sets in x space. Clearly the limit is a travelling wave solution; one could then define the eigenspeeds of a solution u as those values of c for which the above limit exists and is different from zero. It is far from obvious that - as is the case for the KdV equation - the eigenspeeds which appear in the limit $t\to\infty$ and $t\to-\infty$ are the same.

The eigenspeeds c_j are unequivocally determined by the solution u under consideration and thus can be regarded as functionals of solutions. It is clear from the definition that if u is translated by the amounts a and b in the x and t direction, the eigenspeeds remain the same while the phase shifts change by the amount $a - cb$.

We saw earlier that solutions are uniquely determined by their initial values; therefore the eigenspeeds can be also regarded as functionals of the initial values. It follows from translation invariance that the eigenspeeds are invariant functionals (also called integrals); that is, if f and f' denote the value of u at two different times, then $c_j(f) = c_j(f')$. It follows similarly that the difference $\theta_j^+ - \theta_j^-$ is an integral.

This analysis shows that if solutions of the KdV equation behave for large t as indicated in (1.9) then the KdV equation has an infinity of integrals. Indeed, Kruskal, Gardner and Miura, [5], succeeded in constructing explicitly an infinite sequence of integrals; the simplest of these will be described and used in section 2. We present now a general method for constructing an infinite set of integrals for equations of evolutions (1.2).

Let $\mathcal{B}$ be some space of functions chosen so that for each t, $u(t)$ lies in $\mathcal{B}$. Suppose that to each function u in $\mathcal{B}$ we can associate a selfadjoint operator $L = L_u$

over some Hilbert space:

(1.10) $$u \to L_u$$

with the following property: That if u changes with t subject to the equation

$$u_t = K(u)$$

the operators $L(t)$, which also change with t, remain unitarily equivalent. If this is the case then the <u>eigenvalues</u> of L_u constitute a set of integrals for the equation under consideration.

The unitary equivalence of the operators $L(t)$ means that there is a one-parameter family of unitary operators $U(t)$ such that

(1.11) $$U(t)^{-1}L(t)U(t)$$

is independent of t. This fact can be expressed by setting the t derivative of (1.11) equal to zero:

(1.12) $$-U^{-1}U_tULU + U^{-1}L_tU + U^{-1}LU_t = 0$$

A one parameter family of unitary operators satisfies a differential equation of the form

(1.13) $$U_t = BU$$

where $B(t)$ is an antiselfadjoint operator. Conversely, every solution of (1.13) with $B^* = -B$, is a one-parameter family of unitary operators. Substituting (1.13) into (1.12)

we get, after multiplication by U on the left, U^{-1} on the right

$$-BL + L_t + LB = 0$$

which is the same as

$$L_t = BL - LB = [B,L]. \tag{1.14}$$

If u satisfies the equation $u_t = K(u)$ then L_t can be expressed in terms of u, and all that remains to verify is that equation (1.14) has an antisymmetric solution B.

The drawback of this method is that it requires one to guess correctly the relation (1.10) between the function u and the operator L. Now Gardner, Kruskal and Miura have made the remarkable discovery, [1], that the eigenvalues of the Schrödinger operator

$$L = D^2 + u/6 \tag{1.15}$$

are invariant if u varies according to the KdV equation. We shall presently verify this fact with the aid of the linear operator equation (1.14); more generally we shall use equation (1.14) to find a class of differential equations under which the operators (1.15) are unitarily equivalent for all t.

With the choice (1.15) the operator L_t reduces to multiplication by u_t so that according to (1.14) we have to find an antisymmetric operator B whose commutator with L is multiplication. An obvious choice is

$$(1.16)_o \qquad B_o = D$$

Indeed an easy calculation gives

$$[B_o, L] = \frac{1}{6} u_x .$$

This shows that if u varies according to the equation

$$(1.17) \qquad u_t = \frac{1}{6} u_x ,$$

the operators (1.15) are unitarily equivalent. Alas, this is a trivial fact since changing u according to (1.17) amounts to replacing the potential u by a translate of u, which obviously results in an equivalent operator. For a less trivial result we try a third order antisymmetric operator

$$(1.16)_1 \qquad B_1 = D^3 + bD + Db,$$

the coefficient b to be chosen. A brief calculation yields the following value for the commutator:

$$[B_1, D] = \frac{1}{2} u_x D^2 + \frac{1}{2} u_{xx} D + \frac{1}{6} u_{xxx}$$
$$- 4 b_x D^2 - 4 b_{xx} D - b_{xxx} + \frac{1}{3} b u_x$$

Clearly to eliminate all but the zero order terms we have to choose

$$b = \frac{1}{8} u$$

With this choice $[B_1, L]$ is multiplication by

$$\frac{1}{24}[u_{xxx}+uu_x]$$

Multiplying B_1 by -24 we verify that

$$[B_1,D] = K(u)$$

where $K(u) = u_t$ is the KdV equation (1.1)!

Clearly this process can be generalized; we could choose B_q as a skew symmetric differential operator of any odd order $2q + 1$:

$$B_q = D^{2q+1} + \sum_1^q b_j D^{2j-1} + D^{2j-1} b_j$$

Since $[B_q,L]$ is symmetric the requirement that it be of degree zero imposes q conditions; these uniquely determine the q coefficients b_j, and the zero order term of $[B_q,L] = K_q$ determines a higher order KdV equation

$$u_t = K_q(u)$$

which shares with the KdV equation the property that the eigenvalues of the Schrödinger equation with u as potential are its integral.

Gardner has discovered[1] an interesting relation between the higher order KdV equations and the explicit sequence of invariants mentioned earlier; this relation will be describe in section 2.

The process described above can be generalized:

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[1] Personal communication

Theorem 1.1: *Suppose that* L *is a selfadjoint operator depending on* u *in the following fashion*:

$$L_u = L_o + M_u$$

where L_o *is independent of* u *and* M *depends linearly on* u. *Suppose that there exists an antisymmetric operator* $B = B_u$ *such that*

$$[B, L_u] = M_{K(u)}.$$

Then the eigenvalues of L_u *are integrals of*

$$u_t = K(u).$$

As example of this procedure we take u to be a $p \times p$ symmetric matrix variable and take L to be the matrix operator $L = D^2 + u/6$. If we choose B to be a third order matrix operator we obtain the *matrix KdV equation*

$$u_t + \frac{1}{2}(uu_x + u_x u) + u_{xxx} = 0.$$

Other choices for the operator L_u should lead to other classes of equations.

Having shown that the eigenvalues $\lambda_1(u), \ldots, \lambda_N(u)$ Schrödinger operator (1.15) are integrals for the KdV equation one asks how these integrals are related to the eigenspeeds $c_1(u), \ldots, c_N(u)$ which appear in the asymptotic

description (1.9). Gardner and Kruskal have found the answer:

$$c_j(u) = 4\lambda_j(u) . \tag{1.18}$$

We give here a derivation of this result which makes use of a general relation involving integrals of nonlinear equations.

This relation is an extension of the following well-known fact about <u>quadratic</u> integrals Q for <u>linear</u> equations: That the <u>bilinear</u> functional $Q(u,v)$ is independent of t for any <u>pair</u> of solutions u and v. This result can be deduced from the invariance of Q for the superimposed solutions $u + v$ and $u - v$. The corresponding result for integrals of a nonlinear equation

$$u_t = K(u) \tag{1.19}$$

is derived by considering one-parameter families u_ε of solutions; these can be constructed e.g. by making the initial value of $u_\varepsilon(t)$ a function of ε:

$$u_\varepsilon(0) = u_0 + \varepsilon f$$

We assume that the nonlinear operator K depends differentiably on u, i.e. that

$$\frac{d}{d\varepsilon} K(u+\varepsilon v)\bigg|_{\varepsilon=0} = V(u)v \tag{1.20}$$

exists and is a <u>linear</u> function of v. We call the linear operator $V(u)$ the <u>variation</u> of K. Differentiating the equation

$$\frac{d}{dt} u_\varepsilon = K(u_\varepsilon)$$

we obtain the <u>variational equation</u>

(1.21) $$v_t = V(u)v$$

for the quantity

(1.22) $$v = \left.\frac{du_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} .$$

Let $I(u)$ be an integral of the equation under consideration; we assume that $I(u)$ is differentiable in the Frechet sense, i.e. that

$$\left.\frac{d}{d\varepsilon} I(u+\varepsilon v)\right|_{\varepsilon=0}$$

exists and is a <u>linear</u> functional of v. This linear functional can be represented as

(1.23) $$\frac{d}{d\varepsilon} I(u+\varepsilon v) = (G(u),v)$$

where $(\ , \ )$ is some convenient bilinear functional; $G(u)$ is called the <u>gradient</u> of I.

Let $u_\varepsilon(t)$ be the one-parameter family of solutions considered before; then

$$I(u_\varepsilon(t))$$

is independent of t, for every value of ε. Differentiating with respect to ε we obtain that $(G(u),v)$ is independent of t. We formulate this result as

Lemma 1.1: Let $u(t)$ be any solution of equation (1.19), $v(t)$ any solution of the corresponding linear variational equation (1.21). Let I be any integral for (1.19), G its gradient; then

$$(G(u(t)), v(t)) \tag{1.24}$$

is independent of t.

We suppose now that equation (1.19) is translation invariant and that it has a solitary wave solution $s(x-ct)$. We also assume that the bilinear functional $(\ , \ )$ is symmetric and also translation invariant.

Let v be any solution of (1.21); according to (1.24)

$$(G(s(x-ct), v(x,t))$$

is independent of t. Using the translation independence of $(\ , \ )$ and introducing the abbreviation

$$v(x+ct,t) = w(x,t) \tag{1.25}$$

we obtain that

$$(G(s(x)), w(x,t))$$

is equal to the above quantity and therefore also independent of t. Differentiating with respect to t we obtain that

$$(G(s),w_t) = 0. \tag{1.26}$$

From (1.25)

$$w_t = cv_x + v_t;$$

using equation (1.21) and the fact that V commutes with translation we get

$$w_t = [cD+V(s)]w,$$

where D denotes $\partial/\partial x$. Substituting this into (1.26) we get

(1.27) $$(G(s),[cD+V(s)]w) = 0.$$

Denote the adjoint of V by V^*; since $(\ , \ )$ is translation invariant, $D^* = - D$, so we get from (1.27) that

$$([-cD+V^*(s)]G(s),w) = 0.$$

The value of w at any particular time, say $t = 0$ can be prescribed arbitrarily; therefore it follows from the above relation that in fact

(1.28) $$[-cD+V^*(s)]G(s) = 0.$$

Next we make our final assumption about the differential equation (1.19): it is <u>energy preserving</u>, i.e. that for solutions u of (1.19)

$$(u(t),u(t))$$

is independent of t. Differentiating with respect to t and using (1.19) we see that this amounts to requiring that

for all u

$$0 = 2(u,u_t) = 2(u,K(u))$$

The initial value of u is arbitrary. Putting u_ε in place of u we get after differentiating with respect to ε that

$$(v,K(u)) + (u,V(u)v) = 0.$$

Using the adjoint of V this can be written as

$$(v,K(u)+V^*(u)u) = 0.$$

Since v is arbitrary this implies that

(1.29) $$K(u) + V^*(u)u = 0.$$

We turn now to the equation for solitary waves:

$$0 = c\, s_x + K(s)$$

Expressing K from (1.29) we can rewrite this as

$$[cD-V^*(s)]s = 0;$$

i.e. the solitary wave s belongs to the nullspace of the linear operator $cD-V^*(s)$.

We assume now that <u>the only function annihilated by</u> $cD-V^*(s)$ <u>and vanishing at</u> $\pm\infty$ <u>is a multiple of</u> s. From (1.28) we see that $G(s)$ is annihilated by $cD-V^*(s)$; therefore if $G(s)$ vanishes at $\pm\infty$ it follows that

Lemma 1.2: Let G be the gradient of any integral of (1.2); then, under the conditions stated before,

$$G(s) = \kappa s \tag{1.30}$$

i.e. that every solitary wave is an eigenfunction of the gradient of an integral.

It is easy to verify that the KdV equation is energy preserving with respect to the L_2 scalar product. In that case $K(u) = -uu_x - u_{xxx}$, so that

$$V(u)v = -(uv)_x - v_{xxx}$$

and thus

$$\overset{*}{V}(u) = uD + D^3.$$

It is not hard to show that every function annihilated by

$$(C-s)D - D^3$$

and vanishing at $\pm\infty$ is a constant multiple of s.

We apply now the foregoing to the integral $I(u) = \lambda(u)$, where λ is an eigenvalue of $L = D^2 + u/6$:

$$Lw = \lambda w. \tag{1.31}$$

To compute the gradient of λ we replace u by $u + \varepsilon v$ and differentiate with respect to ε. Denoting $\frac{d}{d\varepsilon}$ by a dot and using the fact that $\dot{L} = \dot{u}/6 = v/6$ we get

$$L\dot{w} + vw/6 = \lambda\dot{w} + \dot{\lambda}w.$$

We take the L_2 scalar product with w; using the fact that L is symmetric and that (1.31) is satisfied we get rid of

$\dot{w}$ and obtain

$$(vw/6, w) = \dot{\lambda}(w,w).$$

Assuming that w is normalized so that $(w,w) = 1$ we get that

$$\dot{\lambda} = (vw/6, w) = \int vw^2/6 \, dx = (\frac{w^2}{6}, v).$$

Since $\dot{\lambda} = \frac{d}{d\varepsilon} I = (G(u), v)$, this shows that the gradient G of λ is given by

(1.32) $$G(u) = \frac{w^2}{6} .$$

Since eigenfunctions w vanish at $\pm \infty$ the hypothesis preceding (1.30) is fulfilled; therefore we conclude from (1.30) that $G(s)$ is a constant multiple of s:

$$G(s) = \frac{w^2}{6} = \kappa s,$$

i.e. that the eigenfunction w of

$$L = D^2 + s/6$$

is

(1.33) $$w = \text{const } s^{1/2} .$$

This relation is easily verified by an explicit calculation: Taking the constant to be 1 we get

$$w_x = \frac{1}{2} s_x s^{-1/2}$$

$$w_{xx} = \frac{1}{2} s_{xx} s^{-1/2} - \frac{1}{4} s_x^2 s^{-3/2}$$

Using relations (1.6) and (1.7) for the solitary wave we get

$$w_{xx} = \frac{1}{2}(cs - \frac{s^2}{2})s^{-1/2} + \frac{1}{4}(\frac{s^3}{3} - cs^2).$$

Substituting this into the eigenvalue equation

$$Lw = w_{xx} + \frac{sw}{6}$$

we get after a brief calculation that

$$(1.34) \qquad Lw = Ls^{1/2} = \frac{c}{4}\, s^{1/2}.$$

This proves that

$$c(s) = 4\lambda(s).$$

Let u be any solution of the KdV equation which contains a solitary wave travelling with speed c, i.e. such that given any positive ε and X, there exist T such that

$$(1.35) \qquad |u(x+cT,T)-s(x-\Theta)| < \varepsilon$$

for all $|x| < X$. We claim that then the operator $L_T = D^2 + u(T)$ has $c/4$ for an approximate eigenvalue and

$$w_T(x) = s^{1/2}(x-cT-\Theta)$$

as approximate eigenfunction, in the sense that

$$(1.36) \qquad ||L_T w - \frac{c}{4}\, w|| \le \delta||w||$$

where δ tends to zero as $\varepsilon \to 0$ and $X \to \infty$. To see this we use the fact that according to (1.35) in the interval

(1.37) $$cT - X < x < cT + X$$

u differs by ε from $s(x-cT-\theta)$; therefore using (1.34) we conclude that

(1.38) $$|L_T w_T - \frac{c}{4} w_T| < \varepsilon w_T$$

in the interval (1.37). On the other hand it follows from formula (1.8) that outside of the interval (1.37) w_T and its second derivative are bounded by $\exp(-\text{const}|X-x|)$. Denoting by M the supremum $u(x,t)$ it follows that

(1.38)' $$|L_T w_T - \frac{c}{4} w_T| < (\text{const} + M)e^{-\text{const}|X-x|}$$

outside the interval (1.37). Combining (1.38) and (1.38)' we deduce (1.36).

According to spectral theory inequality (1.36) implies that $c/4$ lies within δ of a point of the spectrum of L_T; since we have seen earlier that the spectrum of L_T is independent of T, it follows that $c/4$ is an eigenvalue of L.

This proves one half of (1.18); the second half - that if λ is an eigenvalue of L_u then 4λ is an eigenspeed of u - has not yet been demonstrated.

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[1] In section 2 we shall demonstrate the uniform boundedness of solutions of KdV equation.

2. In [9] Kruskal and Zabusky have studied the interaction of solitary waves; in particular they posed the following problem:

Let d be a solution of KdV which for t large negative represents two solitary waves travelling with speed c_1, respectively c_2, approaching each other; what is the asymptotic behavior of $d(x,t)$ for large positive values of t?

They solved this problem by computing d; for $-T$ large negative set

$$d(x,-T) = s(x,c_1) + s(x-X,c_2) \tag{2.1}$$

where the separation distance X was chosen so large that the two solitary waves overlap only by a negligible amount; since solitary waves die down exponentially, even a moderate value of X accomplishes this. The speeds were chosen so that $c_1 > c_2$, and X was taken to be positive so that at $t = -T$ the faster wave lies to the left of the slower one.

If the equation governing the motion were linear, the solitary waves would not interact at all and so after the elapse of $2X/(c_1-c_2)$ time the relative position of the two would be merely interchanged. Numerical calculation of the solution of the KdV equation with initial values

(2.1) showed that for $S > 2X/c_1-c_2$

$$d(x,-T+S) = s(x-c_1S-\theta_1,c_1) + s(x-X-c_2S-\theta_2,c_2),$$

except for deviations that could be accounted for by truncation error, i.e. the same as would be in a linear case except for phase shifts θ_1 and θ_2.

The actual process of interaction, i.e., the behavior of $d(x,t)$ around the time $-T + X/c_1-c_2$ is far from being a mere superposition. In fact Kruskal and Zabusky observed that in cases when $c_1 \gg c_2$, i.e. when the first wave was much higher (and therefore faster) than the second one, the big wave swallows up the small one during the interaction, and reemits it later. In cases where c_1 and c_2 were comparable the two waves interact as follows: As soon as the big wave comes reasonably close to the smaller one in front of it the big wave begins to shrink and the smaller one begins to grow, until the two waves interchange their roles; thereafter they separate.

In this section we present a rigorous proof that the KdV equation indeed has solutions which behave like two solitary waves approaching, interacting, then separating; furthermore we give a precise estimate for the ratio of speeds which lead to the two different kinds of interaction described above. It turns out that there is yet a third manner, intermediate between the other two, in which the interaction can

take place.

We shall call these special solutions <u>double waves</u>, and denote them by $d(x,t;c_1,c_2)$. Note that for fixed c_1 and c_2 there is a two-parameter family of double waves, the parameters being the phase of each solitary wave at $t = -\infty$.

In [8] Kruskal and Zabusky derived an ordinary differential equation with respect to x which $d(x,t;c_1 c_2)$ satisfies for each fixed t. The argument in [8] is formal; here we present an (almost) rigorous derivation.

Our starting point is lemma 1.1, the constancy of the functional (1.24) for pairs of solutions u, v of the nonlinear equation and of the variational equation respectively. Taking u to be d we get that for any solution v of (1.21),

$$(G(d),v) \tag{2.2}$$

where G is the gradient of some integral, is independent of t. By definition, for t large negative

$$\begin{aligned} d(x,t;c_1 c_2) = {} & s(x-c_1 t-\theta_1,c_1) \\ & + s(x-c_2 t-\theta_2,c_2) \\ & + \text{error}(t). \end{aligned} \tag{2.3}$$

The error term in (2.3) is due to the interaction of the tails of the solitary waves into which the double wave d

decomposes. Since the tails of solitary waves decay exponentially, and since the separation of the two solitary waves is proportional to t, we expect the error term in (2.3) to decay exponentially in t. Suppose that the gradient G is a local operator; it follows from (2.3) that

$$G(d) = G(s_1) + G(s_2) + \text{error}, \tag{2.4}$$

where the error in (2.4) also tends to zero exponentially as $t \to -\infty$.

Suppose that G satisfies

$$G(s_1) = 0, \ G(s_2) = 0; \tag{2.5}$$

then it follows from (2.4) that $V(d)$ tends to zero exponentially as $t \to -\infty$.

We turn now to the functions v; since these satisfy a linear equation it can be shown, by an argument similar to the one which led to inequality (1.4), that any v increases at most exponentially. In fact since exponentially increasing solutions of the variational equation usually indicate an instability, and since on the other hand numerical evidence indicates that double waves are stable, it is reasonable to expect that all solutions v of the variational equation grows at a rate slower than exponential.

We return now to the functional (2.2); we have shown that as $t \to -\infty$ the first factor $G(d)$ tends

to zero exponentially, and that the second factor v tends to infinity slower than exponentially. It follows then that $(G(d),v)$ tends to zero at $t \to -\infty$; on the other hand according to (2.2) this functional is independent of t. Therefore it follows that $(G(d),v)$ is zero for all t.

At any particular time t_o the initial values of v may be prescribed arbitrarily; therefore it follows that

$$G(d) = 0 \tag{2.6}$$

at time t, that is for anytime.

There remains the task of constructing an integral I whose gradient G is local and satisfies (2.5), i.e. annihilates both solitary waves s_1 and s_2. Here we rely on lemma 1.2, relation (1.30), according to which solitary waves are eigenfunctions of the gradient of <u>every</u> integral:

$$G(s_1) = \varkappa(G)s_1 \quad , \quad G(s_2) = \varkappa(G)s_2$$

It follows that, given three independent integrals, an appropriate linear combination of them will have a gradient which annihilates any two given solitary waves.

We turn now to the task of finding three independent integrals whose gradients are local operators. We have already noted in section 1 that the energy

$$I_1(u) = \int \frac{1}{2} u^2 dx = \frac{1}{2}(u,u) \tag{2.7$_1$}$$

is an integral for the KdV equation. Another integral was found by Whitham, [7]:

$$(2.7)_2 \qquad I_2(u) = \int (\tfrac{1}{3} u^3 - u_x{}^2)dx.$$

A third integral was discovered by Kruskal and Zabusky:

$$(2.7)_3 \qquad I_3(u) = \int (\tfrac{1}{4} u^4 - 3uu_x{}^2 + \tfrac{9}{5} u_{xx}{}^2)dx.$$

Subsequently Kruskal and Zabusky found two more explicit integrals and Miura four more; an infinite sequence[1] of them was constructed in [5].

For the discussion of the double wave we need only the first three integrals. The gradients of these are

$$(2.8)_1 \qquad G_1(u) = u,$$

$$(2.8)_2 \qquad G_2(u) = u^2 + 2u_{xx},$$

$$(2.8)_3 \qquad G_3(u) = u^3 + 3u_x{}^2 + 6uu_{xx} + \tfrac{18}{5} u_{xxxx}.$$

Note that these are indeed local operators. The proof that I_1, I_2, I_3 are integrals of KdV consists in

---

[1] Denote by I_n the n^{th} integral of this sequence and by G_n its gradient. The relation discovered by Gardner between I_n and the n^{th} generalized KdV operator K_n described in section 1 is as follows:

$$K_n = DG_n.$$

verifying that the product of $G_n(u)$ with $K(u) = uu_x + u_{xxx}$ is a perfect x derivative. Indeed an explicit calculation gives

$$(2.9)_1 \qquad G_1(u)K(u) = (uu_{xx} - \tfrac{1}{2} u_x{}^2 + \tfrac{1}{3} u^3)_x = H_1(u)_x$$

$$(2.9)_2 \qquad G_2(u)K(u) = (u_{xx}{}^2 + u^2u_{xx} + \tfrac{1}{4} u^4)_x = H_2(u)_x$$

$$(2.9)_3 \qquad G_3(u)K(u) = (\tfrac{9}{5} u_{xxx}{}^2 + \tfrac{18}{5} u_{xxx}u_xu +$$
$$+ \tfrac{6}{5} u_{xx}{}^2 u - \tfrac{3}{5} u_{xx} u_x{}^2 + \tfrac{3}{2} u_x{}^2u + u_{xx}u^3 + \tfrac{u^5}{5})_x$$
$$= H_3(u)_x \quad .$$

We know that solitary waves are eigenfunctions of G_n; a calculation gives the eigenvalues $\mathcal{K}(G_n)$ as follows:

$$(2.10)_1 \qquad G_1(s) = s$$

$$(2.10)_2 \qquad G_2(s) = 2cs$$

$$(2.10)_3 \qquad G_3(s) = \tfrac{18}{5} c^2s$$

We form now the linear combination

$$(2.11) \qquad I = I_3 + AI_2 + BI_1$$

whose gradient is

$$(2.12) \qquad G = G_3 + AG_2 + BG_1$$

In view of (2.10)

$$G(s) = (\frac{18}{5} c^2 + 2Ac + B)s.$$

Thus G annihilates s_1 and s_2 if c_1 and c_2 satisfy the equation

$$(2.13) \qquad S(c) = \frac{18}{5} c^2 + 2Ac + B = 0.$$

In view of the relation between coefficients and roots of a quadratic equation this means

$$(2.14)_A \qquad A = -\frac{9}{5}(c_1+c_2),$$

$$(2.14)_B \qquad B = \frac{18}{5} c_1 c_2.$$

In view of (2.6) we conclude: Let d be a double wave with speeds c_1 and c_2, and define the constants A, B by (2.14); then for each t, d satisfies

$$(2.15) \qquad G(d) = G_3(d) + AG_2(d) + BG_1(d) = 0$$

(2.15) is a nonlinear ordinary differential equation of fourth order[1]; therefore its solutions form a 4-parameter family of functions; on the other hand the double waves form a 2-parameter family. We shall accordingly deduce from (2.15) a second order equation satisfied by all double waves. To do this we make use of the earlier observation, see (2.9),

[1] This equation appears in [8].

that the invariance of $I(u)$ for solutions of KdV is equivalent with the fact that $G(u)K(u)$ is a perfect x derivative. Multiplying (2.15) by $K(u)$ and using (2.9) we get that

$$(H_3 + AH_2 + BH_1)_x = 0.$$

Integrating this and using the fact that d and all its derivatives are zero at $x = \pm\infty$ we deduce

$$H_3(d) + AH_2(d) + BH_1(d) = 0. \tag{2.16}$$

Next we make use of the <u>translation</u> <u>invariance</u> of the integrals I under consideration, i.e. that $I(u_\varepsilon)$ is independent of ε, where u_ε is the translate of u in the x direction by ε. Differentiating with respect to ε we get

$$0 = \frac{d}{d\varepsilon} I(u_\varepsilon)\Big|_{\varepsilon=0} = \int G(u)u_x \, dx.$$

This implies that $G(u)u_x$ is a perfect x derivative. Indeed an explicit calculation gives

$$(2.17)_1 \qquad G_1(u)u_x = (\tfrac{1}{2} u^2)_x = J_1(u)_x$$

$$(2.17)_2 \qquad G_2(u)u_x = (u_x{}^2 + \frac{u^3}{3})_x = J_2(u)_x$$

$$(2.17)_3 \qquad G_3(u)u_x = (\frac{18}{5} u_x u_{xxx} - \frac{9}{5} u_{xx}{}^2 + 3uu_x{}^2 + \frac{u^4}{4})_x$$

Thus multiplying (2.15) by u_x we get, after integration and using the fact that d and its derivatives

vanish at ∞, that

$$J_3(d) + AJ_2(d) + BJ_1(d) = 0 \tag{2.18}$$

Both (2.16) and (2.18) are third order differential equations. Expressing d_{xxx} from (2.18), substituting into (2.16) and multiplying by $d_x^{\,2}$ we get an equation of second order and fourth degree in d_{xx} which we write symbolically as

$$Q(d_{xx}, d_x, d) = 0. \tag{2.19}$$

From this equation d_{xx} can be expressed as a 4-valued function of d and d_x; since x does not appear explicitly in these equations this second order equation is equivalent to a first order autonomous system of equations for d and d_x. By studying carefully the geometry of all four branches of the corresponding vector-field one can show that (2.19) indeed has solutions which tend to zero as $x \rightarrow \pm \infty$ and has the shape of a double wave, i.e. has two maxima and one minumum. We shall not present the details because

a) We didn't carry them out completely

b) The formulas are horribly complicated

c) An explicit formula for double waves (indeed N-tuple waves) was derived recently in [1].

We show now how to use equation (2.19) to study the time history of double waves. For this purpose we study

the <u>maximum value</u> of $d(x,t)$ with respect to x, or rather the relative maxima of d as functions of time. Denote by $m = m(t)$ the value of a relative maximum of $d(x,t)$; at the point y where the relative maximum occurs

(2.20) $$d_x = 0$$

It follows from (2.20) by the implicit function theorem that if $d_{xx} < 0$ at y, y is a differentiable function of t; then $m = d(y,t)$ also is differentiable and satisfies

$$m_t = d_x y_t + d_t = d_t \quad .$$

Since d satisfies the KdV equation, we have, in view of (2.20)

(2.21) $$m_t = - K(d) = - d_{xxx}.$$

We proceed now to determine d_{xxx} at a local maximum. At a point where $d_x = 0$ equation (2.16) simplifies considerably; using formulas (2.9) and denoting the value of d by m we get

(2.22) $$\frac{9}{5} d_{xxx}^{\ 2} + \frac{6}{5} d_{xx}^{\ 2} m + d_{xx} m^3 + \frac{m^5}{5}$$
$$+ A(d_{xx}^{\ 2} + m^2 d_{xx} + \frac{1}{4} m^4) + B(m d_{xx} + \frac{1}{3} m^3) = 0.$$

Similarly, at a point where $d_x = 0$ equation (2.18) becomes

$$(2.23) \qquad d_{xx}^{\,2} = \frac{5}{9}\left(\frac{m^4}{4} + A\,\frac{m^3}{3} + B\,\frac{m^2}{2}\right) = P(m).$$

From (2.23) we deduce that

$$(2.24) \qquad d_{xx} = -\,P^{1/2}(m);$$

the negative sign is chosen on account of the second derivative being nonpositive at a local maximum point. Substituting (2.24) into (2.22) we get

$$(2.25) \qquad d_{xxx}^{\,2} = R(m)$$

where

$$(2.26) \qquad R(m) = -\,a(m) + b(m)P^{1/2}(m),$$

with

$$(2.27)_a \qquad a(m) = \frac{2}{3}\,mP(m) + \frac{m^5}{9} + \frac{5AP(m)}{9} + \frac{5Am^4}{36} + \frac{5B}{27}\,m^3$$

and

$$(2.27)_b \qquad b(m) = \frac{5}{9}\,m^3 + \frac{5A}{9}\,m^2 + \frac{5}{9}\,Bm\ .$$

Combining (2.21) and (2.25) we have

$$(2.28) \qquad m_t = \pm\,R(m)^{1/2};$$

the sign to be taken as positive when m increases, negative when m decreases. Thus m as function t is governed by equation (2.28).

To study the behavior of m we have to know something

about the function $R(m)$. We turn now to this task; we start with the observation that the solitary waves s_1 and s_2 themselves satisfy the double wave equation, and therefore their maxima satisfy equation (2.28). On the other hand the maximum of s_1, respectively s_2, is independent of t, being equal to

$$(2.29) \qquad m_1 = 3c_1, \text{ respectively } m_2 = 3c_2.$$

This shows that the constants m_1, m_2 satisfy (2.28) which implies that R vanishes at m_1 and m_2. Actually more than this is true:

<u>Lemma 2.1</u> $R(m)$ has a double zero at m_1 and at m_2, and $\frac{d^2R}{dm^2}$ is positive at these points.

<u>Proof</u>: The polynomials P and S, defined by (2.23), respectively (2.13), satisfy

$$(2.30) \qquad P(m) = \frac{5}{18} m^2 S(m/3) + \frac{m^4}{36} .$$

Since $S(c) = 0$ for $c = c_1$ or c_2, it follows by (2.29) and (2.30) that $P(m) = m^4/36$ for $m = m_1$ or m_2, and so for these values

$$P(m)^{1/2} = m^2/6.$$

Using this relation and expressions (2.14) for A and B we can express dR/dm and d^2R/dm^2 at m_1 and m_2 as polynomials in m_1 and m_2. It is then a matter of

explicit calculation, omitted here, to show that the former are zero and the latter positive. This completes the proof of lemma 2.1. It follows that $R(m)$ is positive in some neighborhood of m_1 and m_2; we turn now to the behavior of $R(m)$ in the whole interval (m_1, m_2). First of all we investigate the sign of $P(m)$; since $P(m)$ is $\frac{5}{9} m^2$ times the quadratic polynomial

$$\frac{m^2}{4} + \frac{A}{3} m + \frac{B}{2} , \tag{2.31}$$

it will be positive if the discriminant of (2.31) is:

$$\text{discr} = \frac{B}{2} - \frac{A^2}{9} .$$

Using formulas (2.14) for A and B we get

$$\text{discr} = \frac{9}{25}(3c_1c_2 - c_1^{\,2} - c_2^{\,2}) .$$

This quadratic form is positive if and only if

$$\frac{c_1}{c_2} \leq \frac{3+\sqrt{5}}{2} = 2.62 . \tag{2.32}$$

Thus we have proved

<u>Lemma 2.2</u>: If c_1 and c_2 satisfy (2.32), $P(m)$ is positive for all real values of m. If (2.32) is violated, $P(m)$ is negative in the interval (n_1, n_2),

$$n_{1,2} = \frac{2}{5}[\, m_1 + m_2 \pm (m_1^{\,2} + m_2^{\,2} - 3m_1 m_2)^{1/2}] \tag{2.33}$$

It is easy to verify that the interval (n_1, n_2)

lies inside (m_1,m_2).

We turn now to $R(m)$; adding $a(m)$ to both sides of (2.26) and squaring we get

$$R^2 + 2Ra = b^2P - a^2 = T(m)m^4. \tag{2.34}$$

Clearly T is a polynomial in m of degree 6. It follows from (2.34) that T vanishes wherever R does. Since R has a double zero at each of the points m_1 and m_2, so does T; a large amount of highschool algebra yields the remaining quadratic factor of T and, just as in highschool, that quadratic polynomial can be factored. The resulting factored form of T is

$$T(m) = \text{const}(m-m_1)^2(m-m_2)^2(m-m_1-m_2)(m+m_1-m_2) \tag{2.35}$$

Lemma 2.3: a) In the range

$$c_1/c_2 < \frac{3+\sqrt{5}}{2}$$

the function $R(m)$ is positive in (m_1,m_2)

b) In the range

$$\frac{3+\sqrt{5}}{2} < c_1/c_2 < 3$$

R is positive in (m_1,n_1) and in (m_2,n_2), where n_1, n_2 are defined by (2.33).

c) In the range

$$3 < c_1/c_2$$

R is positive on (m_2, n_2) and on $(m_1, m_1 - m_2)$, and $R(m_1 - m_2) = 0$.

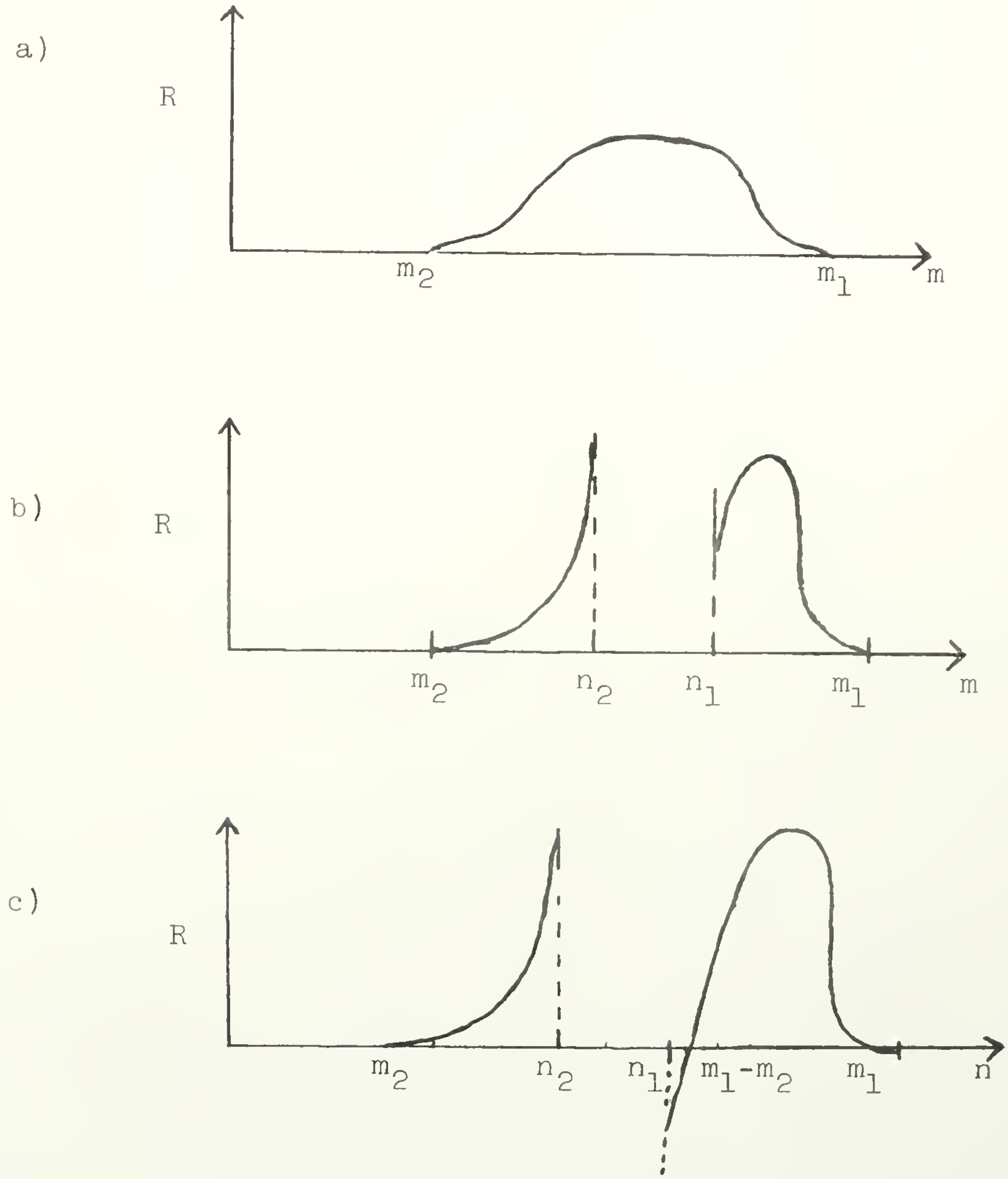

Fig. 1

Proof: As remarked earlier it follows from (2.34) that every root of R is also a root of T. It follows from (2.35) the only possible root of T on (m_1, m_2) is $m = m_1 - m_2$. An explicit substitution into (2.23) and (2.27), using relations (2.14), gives

$$
\begin{aligned}
P(m_1-m_2) &= \frac{1}{36}(m_1-3m_2)^2, \\
a(m) &= \frac{1}{27}(m_1-2m_2)^2(m_1-3m_2), \qquad (2.36)\\
b(m) &= \frac{2}{9}(m_1-2m_2)^2.
\end{aligned}
$$

From the first of these relations we have that

$$
P^{1/2}(m_1-m_2) = \frac{1}{6}\,|m_1-3m_2| = \begin{cases} \frac{1}{6}(3m_2-m_1) & \text{in cases a) and b)} \\ \frac{1}{6}(m_1-3m_2) & \text{in case c)} \end{cases}
$$

Substituting this into the definition (2.26) of R we get, after a short calculation, that

$$
R(m_1-m_2) \quad \begin{cases} \neq 0 & \text{in cases a) and b)} \\ = 0 & \text{in case c),} \end{cases}
$$

as asserted. It is easy to verify that in case c), $m_1 - m_2$ is greater than n_1; this completes the proof of lemma 2.3.

Eventually we will be interested not only in the value of the maximum as function of t but also in its location $y(t)$:

$$d(y(t),t) = m(t).$$

The criterion for a maximum is

$$d_x(y,t) = 0;$$

differentiating this with respect to t we get

$$d_{xx}\, y_t + d_{xt} = 0$$

or

$$y_t = -\frac{d_{xt}}{d_{xx}} . \tag{2.27}$$

Differentiating the KdV equation we get

$$d_{tx} = -d_x^{\,2} - dd_{xx} - d_{xxxx} . \tag{2.28}$$

Using equation (2.15) we can express d_{xxxx} as function of d, d_x, d_{xx}, and d_{xxx}; furthermore, at a local maximum point we have $d_x = 0$, and d_{xx}, d_{xxx} can be expressed as in (2.24) and (2.25) as functions of m. Substituting these expressions into (2.28) and (2.27) we get an expression of the form

$$y_t = Y(m). \tag{2.29}$$

If we know m as function of t, $y(t)$ can be determined by integration.

We shall not calculate $Y(m)$ explicitly; it suffices to note that the discussion applies to the solitary waves s_1 and s_2; for these the value of the maximum is m_1, respectively m_2, and the location of the maximum moves with speed $c_1 = \frac{1}{3}\, m_1$ and $c_2 = \frac{1}{3}\, m_2$ respectively. Therefore (2.29) implies that

$$(2.30) \qquad Y(m_1) = c_1 \;, \quad Y(m_2) = c_2.$$

We are now in a position to prove

Theorem 2.1: For any pair of speeds c_1 and c_2 there exists a double wave, i.e. a solution $d(x,t)$ of the KdV equation such that

$$(2.31) \qquad d(x,t) - s(x-c_1t-\theta_1^{\pm};c_1) - s(x-c_2t-\theta_2^{\pm};c_2)$$

tends to zero uniformly as $t \to \pm\infty$.

Proof: We take as initial value of d a solution (2.19); according to the existence theorem in [6] the KdV equation has a solution $d(x,t)$ for all time with these initial data, and d and all its derivatives are zero at $x = \pm\infty$. We claim that for all values of t, $d(x,t)$ satisfies (2.19). To see this we note first that since $d(x,0)$ satisfies (2.19), it also satisfies (2.15):

$$(2.32) \qquad G(d(0)) = 0.$$

According to lemma 1.1, equation (1.24), for any solution v of the variational equation

$$(2.33) \qquad (G(d),v)$$

is independent of t. Relation (2.32) shows that at $t = 0$, (2.33) is zero; therefore (2.33) is zero for all time. Since the value of v can be prescribed arbitrarily at any particular time, it follows that $G(d)$ is zero at

each time. From this, and the fact that d and its derivatives are zero at $x = \pm\infty$ we deduce as before that d satisfies equation (2.19) at each t.

We turn now to case a) and claim:

For any time t, $d(x,t)$ has exactly two local maxima.

<u>Proof</u>: In case a) the number of local maxima is independent of t since at the time of the creation or disappearance of a local maximum $d_{xx} = 0$; on the other hand, according to (2.23) $d_{xx}^2 = P(m)$ at a local maximum, and in case a) $P(m)$ is positive for all m.

The time history of each local maximum is governed by equation (2.28). In case a) R is positive in (m_1, m_2) and has a double zero at m_1 and at m_2; it follows from this that each solution of (2.28) whose values[1] lie in (m_1, m_2) goes from m_1 to m_2 as t goes from $+\infty$ to $-\infty$, or the other way, depending on the sign in (2.28). Furthermore the approach of $m(t)$ to m_1 or m_2 as $t \to \pm\infty$ is exponential. It follows from this that the function $Y(m)$ occurring in (2.29), (2.30) tends to c_1, respectively c_2 exponentially as $t \to \pm\infty$. Integrating (2.29) we conclude therefore that

$$(2.34) \qquad y(t) - c_{1,2}t$$

---

[1] It is not hard to show that the values of any solution of (2.19) which is zero at $x = \pm\infty$ lie in (m_1, m_2).

tends to a limiting value as $t \to \pm\infty$.

Next we show that as $m(t) \to m_1$ or m_2, the shape of the curve $d(x,t)$ around the maximum point $y(t)$ tends to the shape of a solitary wave, i.e. that

$$\lim_{t \to \pm\infty} d(x-y(t),t) = s_{1,2}(x), \tag{2.35}$$

uniformly on bounded x-intervals. To prove this we merely note that for each fixed t, $s_1(x)$, $s_2(x)$ and $d(x,t)$ all satisfy the fourth order equation (2.15). Furthermore it follows from relations (2.20), (2.24), (2.30) and (2.25) that the Cauchy data of d at $x = y$ for a fourth order ordinary differential equation, i.e. the values of d and of its first three x-derivatives, tend as $t \to \pm\infty$ to the Cauchy data of $s_1(x)$, respectively $s_2(x)$ at $x = 0$. Relation (2.35) follows then from the continuous dependence of solutions on their Cauchy data.

From (2.35) we can deduce our assertion about the number of maxima; for assume that these were more than two. Then at least two of them would tend to the same limit, say m_1, as $t \to \infty$. It follows from (2.34) that the separation of these two locations of maxima tends to a constant; but this is incompatible with relation (2.35) which says that d looks like s_1 centered around _either_ location.

Likewise it is impossible for d to have only one maximum; for then $d(x-y(t),t)$ would be monotonically

decreasing in x on either side of $y(t)$ and so would tend <u>uniformly</u> to one solitary wave as $t \to -\infty$, the other as $t \to +\infty$. We claim that for all t

$$(2.36) \qquad \int_{|x-y(t)|>X} d^2(x,t)\, dx < \varepsilon(X),$$

where $\varepsilon(X)$ tends to zero as $X \to \infty$. If this were so we could deduce that

$$(2.37) \qquad \lim_{t \to \pm\infty} \int d^2(x-y(t),t)\,dx = \begin{matrix} \int s_1{}^2(x)\,dx \\ \\ \int s_2{}^2(x)\,dx. \end{matrix}$$

But this is a contradiction. For, on the one hand, for $c_1 \neq c_2$

$$\int s_1{}^2 dx \neq \int s_2{}^2 dx \ ;$$

on the other hand $\int d^2 dx$ is invariant under both x and t translation, and so the integral on the left side of (2.37) is independent of t; but then the limits as $t \to +\infty$ and $-\infty$ cannot have different values.

To prove (2.36) we merely note that $\int d(x,t)dx = I_o(d)$ is also an integral for the KdV equation, from which (2.36) follows with

$$\varepsilon(X) = s(X),$$

because of the monotonicity of d on either side of $y(t)$.

Having shown that $d(x,t)$ has exactly two maxima it follows easily by the arguments already presented that

as $t \to \pm \infty$, $d(x,t)$ tends uniformly to the superposition of two solitary waves, each travelling at its own speed. This completes the proof of theorem 2.1 in case a).

The analysis presented above shows that the time history of the two maxima is as follows: as t goes from $-\infty$ to $+\infty$ the height of the larger solitary wave decreases from m_1 to m_2, while the height of the smaller one increases from m_2 to m_1. Thus in this case the two solitary waves interchange their roles without passing through each other, as observed by Kruskal and Zabusky in their calculations.

We turn now to case b); here $R(m)$ is not defined inside (n_1,n_2), which means, see equation (2.23), that no relative maximum of $d(x,t)$ can lie in that interval. We assert that the absolute maximum lies above that interval at all times. For suppose on the contrary that at some time the absolute maximum is less than n_2; then, since the absolute maximum is a continuous function of t, and since on the other hand the value of the absolute maximum cannot cross (n_1,n_2), it follows that $d(x,t)$ does not exceed n_2 for any value of x and t. Let $m(t)$ denote a local maximum at time t. As we saw earlier $m(t)$ satisfies the differential equation (2.28): $m_t = \pm R^{1/2}(m)$. In case b) those solutions of this differential equation whose values lie in (m_2,n_2) behave

as follows: depending on the sign in (2.28), $m(t)$ tends with increasing (decreasing) t to n_2:

$$\lim_{t \to t_o} m(t) = n_2 .$$

Denote as before the location of the maximum by $y(t)$, and set

$$\lim_{t \to t_o} y(t) = y_o .$$

It follows from relations (2.20) and (2.24) that

$$d_x(y_o, t_o) = 0, \quad d_{xx}(y_o, t_o) = 0$$

while it follows from (2.25) and the fact that $R(n_2) > 0$ that

$$d_{xxx}(y_o, t_o) > 0.$$

This implies that the function $d(x,t_o)$ does not have a local - much less global - maximum at $x = y_o$. Since the value of d at $x = y_o$ is n_2, it follows that the global maximum of $d(x,t_o)$ exceeds n_2; this contradicts our previous assumption, and so proves the assertion.

Denote by $M(t)$ the absolute maximum of $d(x,t)$; denote by $m(t)$ that local maximum which at, say, $t = 0$ equals $M(0)$ and which satisfies the differential equation (2.28) for local maxima. The function $m(t)$ is either increasing or decreasing; suppose it is increasing, then

since all local maxima satisfy (2.28), it follows that $m(t)$ remains the absolute maximum for $t > 0$; it further follows from $(2.28)_+$ that $m(t)$ tends to m_1 as t tends to $+\infty$, and that for some finite time t_o

$$\lim_{t \to t_o} m(t) = n_1 .$$

When m reaches the value n_1, it ceases to be a local maximum, so that

$$M(t_o) > m(t_o).$$

Denote by t_1 the infimum of those values of t_1 for which $m(t)$ is the absolute maximum; clearly $t_o < t < 0$. At time t_1 there must be at least two points where the absolute maximum is assumed; denote by $n(t)$ that local maximum which is equal to $m(t)$ at $t = t_1$ but is a decreasing function of t; $n(t)$ satisfies equation $(2.28)_-$, and it follows that as $t \to -\infty$, $n(t)$ tends to m_1.

Denote by $y(t)$ the location at time t of the absolute maximum. As before we deduce that

$$\lim_{|t| \to \infty} d(x+y(t),t) = s_1(x),$$

uniformly on bounded x interval.

The analysis presented in the previous case, when applied to this situation, shows that for $|t|$ large there can be at most one additional local maximum; suppose it exists and is located at $z(t)$. Then it follows as above

that

$$\lim_{|t|\to\infty} d(x+z(t),t) = s_2(x),$$

uniformly on bounded x-intervals.

A careful examination of equation (2.28) satisfied by d shows that d does indeed have another local maximum for $|t|$ large; this proves the theorem in case b).

Case c) can be analyzed in a similar fashion. The main difference is that the function $R(m)$ vanishes at $m = m_1 - m_2$; it is easy to show that this causes solutions of $m_t = \pm R^{1/2}(m)$ in the range $m_1 - m_2 \le m(t) < m_1$ to behave in the following fashion:

There is a value t_o such that $m(t_o) = m_1 - m_2$, and m satisfies

$$m_t = \begin{cases} -R^{1/2}(m) & \text{for } t < t_o \\ R^{1/2}(m) & \text{for } t_o < t \end{cases} .$$

In particular $m(t)$ tends to m_1 as $|t|$ tends to ∞.

As in case b) one can show that for $|t|$ large there is exactly one absolute and one local maximum, and that the asymptotic relation (2.31) holds. This completes the proof of theorem 2.1.

The time history of the local maxima is as follows: As t goes from $-\infty$ to ∞, the amplitude of

the smaller solitary wave increases until it reaches the value n_2, at which point the local maximum disappears. In the meanwhile the amplitude of the larger solitary wave decreases steadily; in case c) this amplitude reaches its minimum value m_1-m_2 at some time T and some point X. The double wave is symmetric with respect to this occurrence, i.e.

$$d(X-x,T-t) = d(x,t). \tag{2.38}$$

In case b) the amplitude of the larger solitary wave decreases until it reaches the value n_1, at which point the local maximum disappears. Before this happens however another local maximum is created which starts increasing. Denote by T the time when the two maxima are equal, and denote by X the midpoint between the two local maxima; the double wave satisfies the symmetry relation (2.38).

Speaking qualitatively we might say that in cases b) and c) the big wave first absorbs, then reemits the small wave, and that in case b) the absorption of the small wave raises a secondary peak on the big wave.

Up to a certain point one can give a similar analysis of N-tuple waves. That is,solutions of the KdV equation which as $t \to \pm\infty$ split apart into a superposition of n-solitary waves with speeds $c_1, c_2, \ldots c_N$. Using the first N of the sequence of integrals constructed in [5]

one can derive an ordinary differential equation of order $2N$ with respect to x. An elegant argument of Gardner (personal communication) shows that the generalized KdV operators K_n, $n = 1, \ldots, N$ are integrating factors for this differential operator. In this fashion we can obtain by elimination differential equation of order N for the N-tuple wave, but the resulting equation is too complicated to yield any useful information.

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